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SOME INEQUALITIES FOR POLYNOMIALS WITH A PRESCRIBED ZERO

BY

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ABSTRACT. Let $f(z)$ be a polynomial of degree n. Given that $f(z)$ has a zero on the circle $|z| = \rho$ $(0 < \rho < \infty)$, we estimate $|f(0)|$ and

$$((2\pi)^{-1} \int_0^{2\pi} |f(e^{i\theta})|^2 \, d\theta)^{1/2}$$

in terms of $\max_{|z|=1} |f(z)|$. We also consider some other related problems.

1. **Introduction and statement of results.** If $f(z)$ is holomorphic in $|z| \leq 1$, then

$$|f(0)| \leqslant \left(\frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 \, d\theta\right)^{1/2} \leqslant \max_{|z|=1} |f(z)| \tag{1.1}$$

where equality can hold in the two cases only when $f(z)$ is of constant modulus on $|z| = 1$. Callahan [3] asked what improvement results in the second inequality from supposing that $f(z)$ is a polynomial of degree at most n having a zero on $|z| = 1$. He answered the question by showing that if $f(1) = 0$, then

$$\left(\frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 \, d\theta\right)^{1/2} \leqslant \left(\frac{n}{n+1}\right)^{1/2} \max_{|z|=1} |f(z)|. \tag{1.2}$$

For an extension and an alternative proof of (1.2), see [2, p. 46].

As for the inequality between the first and the last quantities in (1.1), it is known (see [6, p. 364, problem 8.2], and [7]) that

$$1 - \frac{c_1}{n} \leqslant \sup \frac{|f(0)|}{\max_{|z|=1} |f(z)|} \leqslant 1 - \frac{c_2}{n} \tag{1.3}$$

where c_1 and c_2 are constants not depending on n and the supremum is taken over the class of all polynomials $f(z)$ of degree at most n having a zero on $|z| = 1$.

It is natural to ask what happens when $f(z)$ is assumed to be a polynomial

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of degree at most n having a zero of modulus ρ where ρ is any positive number not necessarily equal to 1. In [4, Corollary 3] it was shown that in that case the first inequality in (1.1) can be replaced by

$$|f(0)| \leqslant \left\{\left(\sum_{k=1}^{n} \rho^{2k}\right) \Big/ \left(\sum_{k=0}^{n} \rho^{2k}\right)\right\}^{1/2} \left(\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2\, d\theta\right)^{1/2} \tag{1.4}$$

which is sharp. Here we estimate $|f(0)|$ and $((2\pi)^{-1} \int_0^{2\pi} |f(e^{i\theta})|^2\, d\theta)^{1/2}$ in terms of $\max_{|z|=1} |f(z)|$ given that $f(z)$ is a polynomial of degree at most n having a zero on $|z| = \rho$. We also prove some other results of the same nature.

We start with the observation that the constant c_1 in (1.3) may be taken to be $(\pi^2)/8 \approx 1.23 \cdots$. This value is better than the best value,

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \left|\log\left(1 - \left(\frac{\sin x}{x}\right)^2\right)\right| dx,$$

of c_1 known [7] so far.

THEOREM 1. *If F is the family of all polynomials of degree at most n having a zero on $|z| = 1$, then*

$$1 - \frac{\pi^2}{8n} \leqslant \sup_{f \in F} \frac{|f(0)|}{\max_{|z|=1} |f(z)|}. \tag{1.5}$$

This is fairly close to the best possible value since the known estimate for c_2, which is not sharp, is approximately 1.

In the other direction we have

THEOREM 2. *Let $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial of degree at most n vanishing at*

$$\zeta = \rho e^{i\varphi} \neq \exp\left(i\frac{2\pi}{n+1}\nu\right) \qquad (1 \leqslant \nu \leqslant n,\ \rho \geqslant 0,\ 0 \leqslant \varphi < 2\pi).$$

Then

$$\begin{aligned} |a_0| \leqslant \frac{2\rho}{n+1} \sum_{\nu=1}^{n} \frac{\sin(\pi\nu/(n+1))}{\sqrt{\rho^2 - 2\rho\cos(2\pi\nu/(n+1) - \varphi) + 1}} \\ \cdot \max_{1 \leqslant k \leqslant n} \left|f\left(\exp\left(i\frac{2\pi}{n+1}k\right)\right)\right|. \end{aligned} \tag{1.6}$$

This inequality is best possible for every admissible $\rho e^{i\varphi}$.

THEOREM 3. *Let $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial of degree at most* *n having a zero of modulus* ρ. *Denote by* σ_n *and* τ_n *the smallest positive roots of the equations*

$$x^{n+1} - 2x + 1 = 0$$

and

$$(n+1)x^{n+2} - (n+3)x^{n+1} + (n+1)x - (n-1) = 0$$

respectively. Then

$$|a_0| \leqslant c(\rho) \max_{|z|=1} |f(z)|$$

where

$$c(\rho) = \begin{cases} \rho - \dfrac{1-\rho}{1+\rho}\dfrac{\rho^{n+1}}{1-\rho^n} & \text{if } 0 \leqslant \rho \leqslant \sigma_n, \\[2ex] \rho - \dfrac{1-\rho}{1+\rho}\dfrac{2\rho^{n+2}}{1-\rho^{n+1}} & \text{if } \sigma_n \leqslant \rho \leqslant \tau_n, \\[2ex] \dfrac{n}{n+1}\dfrac{2\rho}{1+\rho} & \text{if } \tau_n \leqslant \rho \leqslant 1, \\[2ex] \rho c(1/\rho) & \text{if } 1 \leqslant \rho. \end{cases} \tag{1.7}$$

REMARK 1. For small values of ρ we obtain

$$c(\rho) = \rho - \rho^{n+1} + O(\rho^{n+2}).$$

The polynomial

$$f(z) = (z - \rho)(\rho^n z^n - 1)/(\rho z - 1)$$

for which

$$|f(0)| = (\rho - \rho^{n+1} + O(\rho^{2n+1})) \max_{|z|=1} |f(z)|$$

shows that (1.7) is essentially best possible for small ρ. The same remark applies to large values of ρ. Theorem 1 gives the degree of precision in the case $\rho = 1$.

The next result generalizes inequality (1.2).

THEOREM 4. *Let* ρ_n *be the (only) root of* $x^{2n+2} - 2x^{2n+1} + 2x - 1 = 0$ *in* $(1, \infty)$. *If* $f(z)$ *is a polynomial of degree at most n having a zero of modulus* ρ, *then*

$$\left(\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2\, d\theta\right)^{1/2} \leqslant \gamma(\rho) \max_{0 \leqslant \theta < 2\pi} |f(e^{i\theta})|$$

where

$$(1.8)\quad \gamma(\rho) = \begin{cases} \left(1 - \dfrac{\rho - 1}{\rho + 1}\,\dfrac{2\rho^n}{\rho^{2n} - 1}\right)^{1/2} & \text{if } 0 \leqslant \rho \leqslant \rho_n^{-1} \text{ or } \rho_n \leqslant \rho < \infty, \\ \left(1 - \dfrac{\rho - 1}{\rho + 1}\,\dfrac{4\rho^{n+1}}{\rho^{2(n+1)} - 1}\right)^{1/2} & \text{if } \rho \neq 1 \text{ and lies in } [\rho_n^{-1}, \rho_n]. \end{cases}$$

REMARK 2. Callahan's estimate (1.2) can be obtained from (1.8) by a limiting process. For small values of ρ we get

$$\gamma(\rho) = 1 - \rho^n + O(\rho^{n+1}).$$

The polynomial

$$f(z) = (z - \rho)(\rho^n z^n - 1)/(\rho z - 1)$$

for which

$$\left(\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2\, d\theta\right)^{1/2} = (1 - \rho^n + O(\rho^{n+1})) \max_{0 \leqslant \theta < 2\pi} |f(e^{i\theta})|$$

shows that (1.8) is essentially best possible for small ρ. The same remark applies to large values of ρ.

We will deduce Theorem 4 from the following two results which may be looked upon as analogues of Theorems 2 and 3 for trigonometric polynomials.

THEOREM 5. *If* $S(\theta) = a_0 + \Sigma_{\nu=1}^n (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta)$ *is a trigonometric polynomial of degree at most n vanishing at the pair of conjugate points* $\theta = \xi \pm i\eta$ $(\eta \neq 0)$, *then*

$$(1.9)\qquad |a_0| \leqslant \left(1 - \frac{e^\eta - 1}{e^\eta + 1}\,\frac{1}{\sinh n\eta}\right) \max_{1 \leqslant k \leqslant 2n-1} \left|S\left(\xi + \frac{k\pi}{n}\right)\right|$$

This inequality (which extends to $\eta = 0$ *by a limiting process) is best possible.*

THEOREM 6. *If* η_n *denotes the positive root of the equation* $\sinh((n + 1)\eta) = 2\sinh(n\eta)$, *then under the hypothesis (and in the notation) of Theorem 5, we have*

$$|a_0| \leqslant \alpha(\eta) \max_{0 \leqslant \theta < 2\pi} |S(\theta)|,$$

where

$$(1.10)\qquad \alpha(\eta) = \begin{cases} 1 - \dfrac{e^\eta - 1}{e^\eta + 1}\,\dfrac{2}{\sinh((n + 1)\eta)} & \text{if } 0 < \eta \leqslant \eta_n, \\ 1 - \dfrac{e^\eta - 1}{e^\eta + 1}\,\dfrac{1}{\sinh(n\eta)} & \text{if } \eta_n \leqslant \eta < \infty. \end{cases}$$

By limiting process we obtain the sharp estimate

$$|a_0| \leqslant (1 - 1/(n+1)) \max_{0 \leqslant \theta < 2\pi} |S(\theta)|,$$

valid for the trigonometric polynomial $S(\theta) = a_0 + \Sigma_{\nu=1}^{n} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta)$ *having a double zero at a real point.*

2. **Lemmas.** Theorem 2 will be deduced from the following:

LEMMA 1. *If* $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ *is a polynomial of degree at most n vanishing at* $z = \zeta$ $(\zeta^{n+1} \neq 1)$, *then for every complex number* λ,

$$a_0 = \frac{1}{n+1} \sum_{\nu=0}^{n} \left(1 - \frac{\lambda}{\zeta \exp(-i2\pi\nu/(n+1)) - 1}\right) f\left(\exp\left(i \frac{2\pi\nu}{n+1}\right)\right). \tag{2.1}$$

PROOF. Since

$$\frac{1}{2\pi} \int_0^{2\pi} e^{i(\mu-k)\theta}\, d\theta = \begin{cases} 0 & \text{if } \mu \neq k, \\ 1 & \text{if } \mu = k, \end{cases}$$

we can write

$$f(z) = \sum_{\mu=0}^{n} a_\mu \left(\sum_{k=0}^{n} z^k \frac{1}{2\pi} \int_0^{2\pi} e^{i(\mu-k)\theta}\, d\theta \right). \tag{2.2}$$

Using the identity

$$\frac{1}{2\pi} \int_0^{2\pi} e^{i(\mu-k)\theta}\, d\theta = \frac{1}{n+1} \sum_{\nu=0}^{n} \exp\left(i \frac{\mu - k}{n+1} 2\pi\nu\right),$$

valid for $|\mu - k| \leqslant n$, and rearranging the sums in (2.2), we obtain

$$f(z) = \frac{z^{n+1} - 1}{n+1} \sum_{\nu=0}^{n} \frac{1}{z \exp(-i2\pi\nu/(n+1)) - 1} f\left(\exp\left(i \frac{2\pi}{n+1} \nu\right)\right) \tag{2.3}$$

$$(z^{n+1} \neq 1).$$

Since $f(\zeta) = 0$ by hypothesis, we have

$$a_0 = f(0) = f(0) - \lambda f(\zeta)/(\zeta^{n+1} - 1).$$

for every complex number λ. Using (2.3) to express $f(0)$ and $f(\zeta)$ in terms of $f(\exp(i2\pi\nu/(n+1)))$ $(\nu = 0, 1, \ldots, n)$ we readily obtain (2.1).

For the proof of Theorem 3 we will also need the second half of the following lemma. The first half of the lemma will not be needed in the present paper and we state it only for completeness.

LEMMA 2. *Let* $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ *be a polynomial of degree at most n vanishing at* $\zeta = \rho > 0$. *Then for every complex number* λ *we have*:

(i) *in the case* $\rho < 1$,

$$a_0 = \frac{1}{2n} \sum_{\nu=0}^{2n-1} (1 - \lambda A_{n,\nu}(\rho)) f(e^{i\nu\pi/n}), \tag{2.4}$$

and

$$\begin{aligned} a_0 = \frac{1}{n+1} \sum_{\nu=0}^{n} \Bigg\{ &(1 - \lambda A_{n+1,2\nu}(\rho)) f\left(\exp\left(i \frac{2\nu\pi}{n+1}\right)\right) \\ &- \lambda A_{n+1,2\nu+1}(\rho) f\left(\exp\left(i \frac{(2\nu+1)\pi}{n+1}\right)\right)\Bigg\}, \end{aligned} \tag{2.5}$$

where

$$A_{n,0}(\rho) = A_{n,2n}(\rho) = (1+\rho)(1-\rho^n)/(1-\rho),$$

$$A_{n,\nu}(\rho) = 1 - \rho^n + 2 \sum_{j=1}^{n-1} (\rho^j - \rho^n) \cos(j\nu\pi/n) > 0 \qquad (1 \leqslant \nu \leqslant 2n-1);$$

(ii) *in the case* $\rho > 1$,

$$a_0 = \frac{1}{2n} \sum_{\nu=0}^{2n-1} (1 - (-1)^\nu \lambda B_{n,\nu}(\rho)) f(e^{i\nu\pi/n}), \tag{2.6}$$

and

$$\begin{aligned} a_0 = \frac{1}{n+1} \sum_{\nu=0}^{n} \Bigg\{ &(1 - \lambda B_{n+1,2\nu}(\rho)) f\left(\exp\left(i \frac{2\nu\pi}{n+1}\right)\right) \\ &+ \lambda B_{n+1,2\nu+1}(\rho) f\left(\exp\left(i \frac{(2\nu+1)\pi}{n+1}\right)\right)\Bigg\}, \end{aligned} \tag{2.7}$$

where

$$B_{n,\nu}(\rho) = \rho^n A_{n,\nu}(1/\rho) \qquad (0 \leqslant \nu \leqslant 2n).$$

PROOF. If $f(z)$ is a polynomial of degree n, then by a formula in [4, Lemma 1] we have (in our notation) for $R > 1$ and $m \geqslant n$,

$$f(Re^{i\varphi}) = \frac{1}{2m} \sum_{\mu=0}^{2m-1} (-1)^\mu B_{m,\mu}(R) f(e^{i(\varphi+\mu\pi/m)}). \tag{2.8}$$

For $R < 1$ we consider $g(z) = z^m \overline{f(1/\bar{z})}$ and obtain

$$g\left(\frac{1}{R}\, e^{i\varphi}\right) = \frac{1}{2m} \sum_{\mu=0}^{2m-1} B_{m,\mu}\left(\frac{1}{R}\right) e^{im\varphi}\, \overline{f(e^{i(\varphi+\mu\pi/m)})},$$

i.e.

$$f(Re^{i\varphi}) = \frac{1}{2m} \sum_{\mu=0}^{2m-1} R^m B_{m,\mu}\left(\frac{1}{R}\right) f(e^{i(\varphi+\mu\pi/m)}). \tag{2.9}$$

Trivially, or as a special case of (2.3), we have for every $k \geqslant n + 1$,

$$f(0) = \frac{1}{k} \sum_{\nu=0}^{k-1} f(e^{i2\nu\pi/k}). \tag{2.10}$$

Now put $m = n$, $k = 2n$, $R = \rho$, $\varphi = 0$ and combine (2.10) linearly with (2.8) and (2.9), respectively, to get (2.6) and (2.4). In order to get the other formulae of Lemma 2, let $m = n + 1$, $k = n + 1$, $R = \rho$, $\varphi = 0$ and again use (2.8), (2.9) and (2.10).

The following lemma will be needed for the proof of Theorem 5.

LEMMA 3. *Let $S(\theta) = a_0 + \Sigma_{\nu=1}^{n}\{a_\nu \cos(\nu\theta) + b_\nu \sin(\nu\theta)\}$ be a trigonometric polynomial of degree at most n vanishing at the pair of conjugate points $\theta = \xi \pm i\eta$ $(\eta \neq 0)$. Then for every complex number λ, we have*

$$a_0 = \frac{1}{2n} \sum_{k=0}^{2n-1} \left(1 - \lambda \frac{(-1)^k}{e^{2\eta} - 2e^{\eta}\cos(k\pi/n) + 1}\right) S\left(\xi + \frac{k\pi}{n}\right), \tag{2.11}$$

as well as

$$\begin{aligned} a_0 = \frac{1}{n+1} \sum_{k=0}^{n} \Bigg\{ &\left(1 - \frac{\lambda}{e^{2\eta} - 2e^{\eta}\cos(2k\pi/(n+1)) + 1}\right) S\left(\xi + \frac{2k\pi}{n+1}\right) \\ &+ \frac{\lambda}{e^{2\eta} - 2e^{\eta}\cos((2k+1)\pi/(n+1)) + 1} \\ &\cdot S\left(\xi + \frac{(2k+1)\pi}{n+1}\right)\Bigg\}. \end{aligned} \tag{2.12}$$

PROOF. The function $S(z) = a_0 + \Sigma_{\nu=1}^{n}\{a_\nu \cos(\nu z) + b_\nu \sin(\nu z)\}$ is an entire function of exponential type n, which is periodic with period 2π and, hence, bounded on the real axis. By an interpolation formula of Boas [1, p. 30] we have for every real ω,

$$e^{-i\omega} S(x + iy) + e^{i\omega} S(x - iy) = 2 \sum_{k=-\infty}^{\infty} (-1)^k c_k S\left(x - s + \frac{k\pi}{n}\right),$$

where

$$c_k = \frac{ny \operatorname{Im}\{e^{-i\omega} \sin(sn + iyn)\}}{(k\pi - s)^2 + n^2 y^2}$$

and

$$sn = \arg\{\cos(\omega + iny)\}.$$

Thus, if $\omega = 0$, then $s = 0$ and

$$c_k = \frac{ny}{(k\pi)^2 + (ny)^2} \sinh(ny).$$

Since $S(x + k\pi/n) = S(x + (k + 2nj)\pi/n)$ for all integral values of j, we get

$$S(x + iy) + S(x - iy) = 2ny \sinh(ny) \sum_{k=0}^{2n-1} (-1)^k d_k S\left(x + \frac{k\pi}{n}\right),$$

where

$$d_k = \sum_{j=-\infty}^{\infty} \frac{1}{(k + 2nj)^2\pi^2 + (ny)^2}.$$

Decomposing into partial fractions, we obtain

$$d_k = \frac{1}{(2n\pi)^2(\zeta_1 - \zeta_2)} \sum_{j=-\infty}^{\infty} \left(\frac{1}{j - \zeta_1} - \frac{1}{j - \zeta_2}\right),$$

where

$$\zeta_1 = (-k\pi + iny)/2n\pi, \quad \zeta_2 = \bar{\zeta}_1.$$

Comparing with the Mittag-Leffler series of the cotangent function, we find that

$$d_k = \frac{i}{(2n)^2 y} \{\cot(n\zeta_1) - \cot(n\zeta_2)\}$$

$$= \frac{1}{2n^2 y} \frac{e^{2y} - 1}{e^{2y} - 2e^y \cos(k\pi/n) + 1}.$$

Thus, we get the formula

$$\frac{S(x + iy) + S(x - iy)}{2} = (e^{2y} - 1) \sinh(ny) \cdot \frac{1}{2n} \sum_{k=0}^{2n-1} \frac{(-1)^k}{e^{2y} - 2e^y \cos(k\pi/n) + 1} S\left(x + \frac{k\pi}{n}\right), \tag{2.13}$$

which holds as well if n is replaced by any larger integer, for a trigonometric polynomial of degree at most n is also a trigonometric polynomial of degree at most N for every integer $N \geqslant n$. Since for $x = \xi$, $y = \eta$, the left-hand side of (2.13) is zero by hypothesis, we obtain, in particular,

$$0 = \frac{1}{2n} \sum_{k=0}^{2n-1} \frac{(-1)^k}{e^{2\eta} - 2e^\eta \cos(k\pi/n) + 1} S\left(\xi + \frac{k\pi}{n}\right), \tag{2.13$'$}$$

$$0 = \frac{1}{n+1} \sum_{k=0}^{2n+1} \frac{(-1)^k}{e^{2\eta} - 2e^\eta \cos(k\pi/(n+1)) + 1} S\left(\xi + \frac{k\pi}{n+1}\right). \tag{2.13$''$}$$

Furthermore, for every $m \geqslant n + 1$ and all real x, we have

$$a_0 = \frac{1}{m} \sum_{k=0}^{m-1} S\left(x + \frac{2\pi k}{m}\right). \tag{2.14}$$

Now take $m = 2n$, $x = \xi$ in (2.14) and subtract from the right-hand side the quantity

$$\frac{1}{2n} \sum_{k=0}^{2n-1} \lambda \frac{(-1)^k}{e^{2\eta} - 2e^{\eta} \cos(k\pi/n) + 1} S\left(\xi + \frac{k\pi}{n}\right)$$

which is zero according to (2.13′) to get (2.11). In order to prove (2.12) take $m = n + 1, x = \xi$ in (2.14) and subtract from the right-hand side the quantity

$$\frac{1}{n+1} \sum_{k=0}^{2n+1} \lambda \frac{(-1)^k}{e^{2\eta} - 2e^{\eta} \cos(k\pi/(n+1)) + 1} S\left(\xi + \frac{k\pi}{n+1}\right)$$

which is zero according to (2.13″).

3. **Proofs of the theorems.**

Proof of Theorem 1. Let $T_k(x) = \cos(k \arccos x)$ be the Chebyshev polynomial of degree k and consider

$$P(x) = T_k(\sin^2(\pi/4k) + x \cos^2(\pi/4k)).$$

The polynomial $P(x)$ has all its zeros on the unit interval and -1 is one of its zeros. Hence,

$$e^{ik\theta} P(\cos\theta) \equiv g(e^{i\theta}),$$

where

$$g(z) = \tfrac{1}{2}(\cos(\pi/4k))^{2k} + k(\cos(\pi/4k))^{2k-2}(\sin(\pi/4k))^2 z + \cdots$$

is a polynomial of degree $2k$ having all its zeros on $|z| = 1$ and a double zero at $z = -1$. Besides, $\max|g(z)|$ on $|z| = 1$ is equal to 1. By a theorem of Lax [5],

$$\max_{|z|=1} |g'(z)| \leqslant k.$$

Now consider the polynomial $f^*(z) = z^{2k-1}g'(1/z)$. It is clear that $|f^*(z)| \leqslant k$ on $|z| = 1$ and $f^*(-1) = 0$. Since $f^*(0) = k(\cos(\pi/4k))^{2k}$, we conclude that

$$1 - \frac{\pi^2}{16k} + \frac{\pi^4}{6144k^3} < (\cos(\pi/4k))^{2k} \leqslant \sup \frac{|f(0)|}{\max_{|z|=1} |f(z)|},$$

where the supremum is taken over all polynomials of degree $2k - 1$ having a zero on $|z| = 1$. From this (1.5) follows for odd degree n.

That (1.5) holds for even n $(=2k)$ can be seen by considering the polynomial

$$h(z) = f^*(z) + \frac{\pi^4}{6144n^2}(z+1)z^{2k-1}$$

which is of degree $n = 2k$. Besides,

$$h(0) = f^*(0) \quad \text{and} \quad \max_{|z|=1} |h(z)| \leqslant k + \frac{\pi^4}{3072n^2},$$

so that

$$\frac{|h(0)|}{\max_{|z|=1} |h(z)|} \geqslant \frac{(\cos \pi/2n)^n}{1 + \pi^4/1536n^3} > 1 - \frac{\pi^2}{8n}.$$

REMARK 3. Starting with $g(z^m)$, the above idea of proof shows that for n tending to infinity,

$$1 + o(1) \leqslant \sup \frac{|f(0)|}{\max_{|z|=1} |f(z)|}$$

if the supremum is taken over all polynomials of degree n having m zeros on $|z| = 1$ where $\limsup_{n \to \infty} (m/n) = 0$.

PROOF OF THEOREM 2. Putting $\zeta = \rho e^{i\varphi}$ and $\lambda = \rho e^{i\varphi} - 1$ in (2.1) we obtain

$$|a_0| = \frac{1}{n+1} \left| \sum_{\nu=1}^{n} c_\nu f\left(\exp\left(i \frac{2\pi}{n+1} \nu\right)\right) \right|$$

$$\leqslant \frac{1}{n+1} \sum_{\nu=1}^{n} |c_\nu| \max_{1 \leqslant \nu \leqslant n} \left| f\left(\exp\left(i \frac{2\pi}{n+1} \nu\right)\right) \right|,$$

where

$$c_\nu = 1 - \frac{\rho e^{i\varphi} - 1}{\rho \exp(i(\varphi - 2\pi\nu/(n+1))) - 1}.$$

A straightforward calculation shows that

$$|c_\nu| = \frac{2\rho \sin(\pi\nu/(n+1))}{(\rho^2 - 2\rho \cos(2\pi\nu/(n+1) - \varphi) + 1)^{1/2}}.$$

By the Lagrange interpolation formula there exists a polynomial of degree at most n vanishing at $\rho e^{i\varphi}$ and assuming arbitrarily prescribed values at $\exp(i2\pi\nu/(n+1))$ $(1 \leqslant \nu \leqslant n)$. Hence, we can construct a polynomial for which equality in (1.6) is attained.

PROOF OF THEOREM 3. Let $f(\rho) = 0$ $(\rho > 0)$ and put $M = \max_{|z|=1} |f(z)|$.

From inequality (1.6) we can readily deduce that

$$|a_0| \leqslant \frac{n}{n+1}\,\frac{2\rho}{\rho+1}M \qquad (\rho > 0). \tag{3.1}$$

Now let us refer to Lemma 2. It is easily seen that

$$B_{m,0}(\rho) \geqslant B_{m,\mu}(\rho) \qquad (\rho > 1,\ 0 \leqslant \mu \leqslant 2m).$$

Hence, putting λ equal to $1/B_{n,0}(\rho)$, $1/B_{n+1,0}(\rho)$ in (2.6), (2.7), respectively, we obtain on the one hand

$$\begin{aligned} |a_0| &\leqslant \frac{M}{2n}\sum_{\nu=0}^{2n-1}\left(1-(-1)^{\nu}\frac{B_{n,\nu}(\rho)}{B_{n,0}(\rho)}\right) \\ &= M\left\{1-\frac{1}{B_{n,0}(\rho)}\left(\frac{1}{2n}\sum_{\nu=0}^{2n-1}(-1)^{\nu}B_{n,\nu}(\rho)\right)\right\}, \end{aligned} \tag{3.2}$$

and on the other

$$|a_0| \leqslant \frac{M}{n+1}\sum_{\nu=0}^{n}\left\{\left(1-\frac{B_{n+1,2\nu}(\rho)}{B_{n+1,0}(\rho)}\right)+\frac{B_{n+1,2\nu+1}(\rho)}{B_{n+1,0}(\rho)}\right\}. \tag{3.3}$$

Applying (2.8) with $m = n$ to $f(z) \equiv 1$ we obtain

$$\frac{1}{2n}\sum_{\nu=0}^{2n-1}(-1)^{\nu}B_{n,\nu}(\rho) = 1,$$

so that (3.2) becomes

$$|a_0| \leqslant \left(1-\frac{\rho-1}{\rho+1}\,\frac{1}{\rho^n-1}\right)M \qquad (1 < \rho < \infty). \tag{3.4}$$

Again applying (2.8) with $m = n+1$, $\varphi = 0$ to the functions $z^{n+1}+1$, $z^{n+1}-1$, respectively, we get

$$\frac{1}{n+1}\sum_{\nu=0}^{n}B_{n+1,2\nu}(\rho) = \rho^{n+1}+1,$$

$$\frac{1}{n+1}\sum_{\nu=0}^{n}B_{n+1,2\nu+1}(\rho) = \rho^{n+1}-1,$$

which may be substituted in (3.3) to give us

$$|a_0| \leqslant \left(1-\frac{\rho-1}{\rho+1}\,\frac{2}{\rho^{n+1}-1}\right)M \qquad (1 < \rho < \infty). \tag{3.5}$$

Applying (3.4), (3.5), respectively, to the polynomial $(\rho z - 1)f(z)/(z - \rho)$, we obtain

$$|a_0| \leqslant \left(\rho - \frac{1-\rho}{1+\rho}\,\frac{\rho^{n+1}}{1-\rho^n}\right)M \quad (0 < \rho < 1), \tag{3.6}$$

$$|a_0| \leqslant \left(\rho - \frac{1-\rho}{1+\rho}\,\frac{2\rho^{n+2}}{1-\rho^{n+1}}\right)M \quad (0 < \rho < 1). \tag{3.7}$$

In case $\rho < 1$ we have three different estimates for $|a_0|$ at our disposal, namely (3.1), (3.6) and (3.7). Similarly, each of the three inequalities (3.1), (3.4) and (3.5) provides us with an estimate for $|a_0|$ if $\rho > 1$. All these estimates can be extended to the case $\rho = 1$ by continuity. Comparing the various estimates for $|a_0|$ and picking out the best for a given ρ we obtain the result stated in Theorem 3.

PROOF OF THEOREM 5. Applying identity (2.13) to $\frac{1}{2}(e^{inz} + 1)$, $\frac{1}{2}(e^{inz} - 1)$, respectively, and putting $x = 0$, $y = \eta$ we obtain

$$\frac{1}{2n}\sum_{k=0}^{n-1} \frac{1}{e^{2\eta} - 2e^{\eta}\cos(2k\pi/n) + 1} = \frac{1}{2}\,\frac{\cosh(n\eta) + 1}{(e^{2\eta} - 1)\sinh(n\eta)}, \tag{3.8}$$

$$\frac{1}{2n}\sum_{k=0}^{n-1} \frac{1}{e^{2\eta} - 2e^{\eta}\cos((2k+1)\pi/n) + 1} = \frac{1}{2}\,\frac{\cosh(n\eta) - 1}{(e^{2\eta} - 1)\sinh(n\eta)}. \tag{3.9}$$

Now (1.9) is easily obtained if we choose λ equal to $(e^{\eta} - 1)^2$ in (2.11) and then make use of the summation formulae (3.8), (3.9).

As is known from interpolation theory, there exists a trigonometric polynomial of degree n vanishing at the points $\xi \pm i\eta$ and assuming the value 1 at $\xi + k\pi/n$ $(1 \leqslant k \leqslant 2n - 1)$. It is easy to see that for this trigonometric polynomial, the sign of equality in (1.9) will be obtained.

PROOF OF THEOREM 6. We can always replace $\max_{1 \leqslant k \leqslant 2n-1} |S(\xi + k\pi/n)|$ in (1.9) by $\max_{0 \leqslant \theta < 2\pi} |S(\theta)|$. But we can improve upon this bound if $0 < \eta \leqslant \eta_n$ by putting $\lambda = (e^{\eta} - 1)^2$ in (2.12) and estimating $|a_0|$ with the help of (3.8) and (3.9).

PROOF OF THEOREM 4. Note that $S(\theta) = |f(e^{i\theta})|^2$ is a real trigonometric polynomial of degree at most n having a pair of conjugate zeros with imaginary parts $\pm \log \rho$. Furthermore, the constant term of $S(\theta)$ is $(2\pi)^{-1}\int_0^{2\pi} |f(e^{i\theta})|^2\, d\theta$. Hence, the result follows from Theorem 5.

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